

THE RIEMANN-HURWITZ FORMULA

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ABSTRACT. These notes were prepared by Yiming Wang for the talk *The Riemann-Hurwitz formula*, given as part of the seminar *Learning algebraic geometry via curves*, organized by Andrea Panontin and Pavel Sechin during the winter semester 2026 – 2027 at the University of Regensburg.

These notes closely follow [Har77, Section 4.2]. The perspective on scheme theory adopted in these notes follows that of [Hoy26]. Throughout these notes, we will work over an algebraically closed field k of characteristic $\neq 2$, a *curve* over k means a smooth, proper, geometrically integral k -scheme of dimension one, unless otherwise stated, and a *point* on a curve means a *closed point* on the curve, unless otherwise stated. Furthermore, the author imposes finiteness assumptions throughout the text; in our setting, this is equivalent to assuming that the morphism is nonconstant.

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1. THE RIEMANN-HURWITZ FORMULA

Proposition 1.1 (The fundamental exact sequence). *Let $f: Y \rightarrow X$ be a finite, generically separable morphism of curves over k . There exists a canonical exact sequence*

$$0 \rightarrow f^* \Omega_{X/k} \xrightarrow{df} \Omega_{Y/k} \rightarrow \Omega_f \rightarrow 0.$$

Proof. Consider the fundamental exact sequence associated to the morphism $f: Y \rightarrow X$ over $\text{Spec}(k)$.

$$f^* \Omega_{X/k} \xrightarrow{df} \Omega_{Y/k} \rightarrow \Omega_f \rightarrow 0.$$

Since Y and X are smooth over k , the characterizations of smoothness ([Hoy26, Theorem 9.4]) imply that $\Omega_{Y/k}$ and $f^* \Omega_{X/k}$ are line bundles over Y . By taking the pullback along the generic point, $\eta_Y^* (\Omega_{Y/k})$ is identified with $\Omega_{\kappa(Y)/k}$ (see [Hoy26, Proposition 9.42]). Since $f: Y \rightarrow X$ is dominant, the following diagram commutes:

$$\begin{array}{ccc} \text{Spec}(\kappa(Y)) & \xrightarrow{\eta_Y} & Y \\ \downarrow & & \downarrow f \\ \text{Spec}(\kappa(X)) & \xrightarrow{\eta_X} & X. \end{array}$$

Therefore, $\eta_Y^* f^* \Omega_{X/k} \simeq \Omega_{\kappa(X)/k} \otimes_{\kappa(X)} \kappa(Y)$. Under these identifications, the morphism $\eta_Y^* df$ fits into the fundamental exact sequence induced by $k \subseteq \kappa(X) \subseteq \kappa(Y)$ as follows:

$$\Omega_{\kappa(X)/k} \otimes_{\kappa(X)} \kappa(Y) \rightarrow \Omega_{\kappa(Y)/k} \rightarrow \Omega_{\kappa(Y)/\kappa(X)} \rightarrow 0.$$

Since $\kappa(Y)/\kappa(X)$ is a finite separable extension of fields, it is étale and $\Omega_{\kappa(Y)/\kappa(X)} = 0$. Therefore, $\eta_Y^* df$ is a surjective morphism of 1-dimensional $\kappa(Y)$ -vector spaces, hence is an isomorphism.

We will now show that $df: f^*\Omega_{X/k} \rightarrow \Omega_{Y/k}$ is a monomorphism in Mod_Y . Let $(U_i \subseteq Y)_{i \in I}$ be an affine open covering of Y . It suffices to show that $df|_U$ is injective. After refining the open cover by principal affine opens, we may assume that $df|_U$ is given by multiplication by some element $a \in \Gamma(U, \mathcal{O}_U)$. The induced morphism on fraction fields $a: \kappa(Y) \rightarrow \kappa(Y)$ is a $\kappa(Y)$ -linear isomorphism, hence a is nonzero. For an integral domain, multiplication by a nonzero element is injective, hence $df: f^*\Omega_{X/k} \rightarrow \Omega_{Y/k}$ is a monomorphism. $\square$

Remark 1.2. In [Proposition 1.1](#), the key point is that df is nonzero if and only if its restriction to the generic point is nonzero, and this is equivalent to $f: Y \rightarrow X$ being generically separable.

Definition 1.3 (Ramification divisor). Let $f: Y \rightarrow X$ be a finite, generically separable morphism of curves over k . Let $L_f = f^*\Omega_{X/k} \otimes \Omega_{Y/k}^\vee$. The effective Cartier divisor $s_f: L_f \hookrightarrow \mathcal{O}_Y$ is the *ramification divisor* of f .¹

For a finite morphism $f: Y \rightarrow X$ between curves, the *Riemann–Hurwitz formula* relates the genera of Y and X to the ramification of f . We begin by making the notion of ramification precise.

Definition 1.4 (Ramification index). Let $f: Y \rightarrow X$ be a finite morphism of curves, $y \in Y$ and $x = f(y)$, with the induced morphism on stalks $f_y: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$. Then $f_y(t_x) = ut_y^{e_y}$ for some $e_y \in \mathbb{N}_{\geq 1}$,² where t_y and t_x are chosen uniformizers of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$, and $u \in \mathcal{O}_{Y,y}^\times$. The number e_y is the *ramification index* of f at y .

Definition 1.5 (Ramification). Let $f: Y \rightarrow X$ be a finite morphism of curves, $y \in Y$ and $x = f(y)$.

- (a) f is *ramified* at y and x is a *branch point* of f , if $e_y > 1$;
- (b) f is *unramified* at y , if $e_y = 1$;
- (c) the ramification of f at y is *tame*, if $\text{char}(k) = 0$, or $\text{char}(k) = p$ and $p \nmid e_y$;
- (d) the ramification of f at y is *wild*, if $\text{char}(k) = p$ and $p \mid e_y$.

From [Definition 1.5](#), we deduce another characterization of unramifiedness:

Remark 1.6. Let $f: Y \rightarrow X$ be a finite morphism of curves over k , $y \in Y$ and $x = f(y)$. Then f is unramified at y in the sense of [Definition 1.5](#) if and only if it is unramified in the scheme-theoretic sense.

Indeed, the stalk of the quasi-coherent sheaf corresponding to $\Omega_{Y/X}$ at y is computed as $\Omega_{\mathcal{O}_{Y,y}/\mathcal{O}_{X,x}}$. By Nakayama's lemma, it is zero if and only if $\Omega_{\mathcal{O}_{Y,y}/\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$ is zero. By applying $-\otimes_{\mathcal{O}_{Y,y}} \kappa(y)$ to the fundamental exact sequence associated to $k \rightarrow \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$, we obtain the following exact sequence:

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \otimes_{\kappa(x)} \kappa(y) \xrightarrow{\bar{f}} \mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow \Omega_{\mathcal{O}_{Y,y}/\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{Y,y}} \kappa(y) \rightarrow 0,$$

as $\Omega_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \simeq \Omega_{X/k}(x) \simeq \mathfrak{m}_x/\mathfrak{m}_x^2$ (see [\[Hoy26, Remark 9.59\]](#)). The $\kappa(y)$ -linear morphism $\bar{f}$ is given by $\bar{f}([t_x] \otimes 1) = [ut_y]$, whenever $e_y = 1$, and zero otherwise. Therefore, $\text{coker}(\bar{f}) = 0$ if and only if $e_y = 1$.

Remark 1.7 (Cotangent module of a curve). Let Y be a curve over k and $y \in Y$. The cotangent module $\Omega_{\mathcal{O}_{Y,y}/k}$ is a free $\mathcal{O}_{Y,y}$ -module of rank 1, generated by dt_y .

Indeed, $\Omega_{Y/k}$ is a line bundle over Y , hence the pullback bundle $\Omega_{\mathcal{O}_{Y,y}/k}$ is a line over the local ring $\mathcal{O}_{Y,y}$, hence is a free of rank 1. Under the canonical isomorphism

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \simeq \Omega_{\mathcal{O}_{Y,y}/k} \otimes_{\mathcal{O}_{Y,y}} \kappa(y),$$

the class of t_y maps to the class of dt_y . Nakayama's lemma shows that $\Omega_{\mathcal{O}_{Y,y}/k}$ is generated by dt_y .

Indeed, the ramification divisor records the ramification in the following precise sense:

Proposition 1.8. Let $f: Y \rightarrow X$ be a finite, generically separable morphism of curves over k , $y \in Y$ and $x = f(y) \in X$.

¹Classically, the line bundle associated to the effective Cartier divisor $s: L \hookrightarrow \mathcal{O}_X$ is $L^\vee$, corresponding to the dual $s^\vee$.

²Since $f: Y \rightarrow X$ is finite, hence dominant, e_y is nonzero.

- (a) $\text{mult}_y(R_f) = \text{length}_{\mathcal{O}_{Y,y}}((\Omega_f)_y)$;
- (b) if f is tamely ramified at y , then $\text{mult}_y(R_f) = e_y - 1$ and
- (c) if f is wildly ramified at y , then $\text{mult}_y(R_f) > e_y - 1$.
- (d) The support $|R_f|$ is the set of ramification points of f .

Proof. Let t_y and t_x be uniformizers of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ respectively.

- (a) The fundamental exact sequence associated to $Y \rightarrow X \rightarrow \text{Spec}(k)$ implies that

$$0 \rightarrow \mathcal{O}_{Y,y} dt_x \xrightarrow{(df)_y} \mathcal{O}_{Y,y} dt_y \rightarrow (\Omega_f)_y \rightarrow 0$$

is exact (Remark 1.7), where $(df)_y$ is induced by f_y , hence is given by $dt_x \mapsto d(f_y(t_x)) = c_y dt_y$ for some $c_y \in \mathcal{O}_{Y,y}$. By the same virtue as Proposition 1.1, c_y is nonzero, which justifies that the first map is injective. Therefore, $(\Omega_f)_y \simeq \mathcal{O}_{Y,y}/(c_y)$. Since the generator $dt_x \otimes (dt_y)^\vee$ of $(L_f)_y$ is sent by s_f to c_y , the pullback of R_f to $\text{Spec}(\mathcal{O}_{Y,y})$ is defined by (c_y) . Consequently, both sides in (a) are equal to $v_y(c_y)$ ([Hoy26, Construction 10.82]).

- (b) Suppose that $f_y(t_x) = ut_y^{e_y}$ for some $u \in \mathcal{O}_{Y,y}^\times$ and $du = w dt_y$ for some $w \in \mathcal{O}_{Y,y}$. By the Leibniz rule,

$$d(f_y(t_x)) = d(ut_y^{e_y}) = wt_y^{e_y} dt_y + ue_y t_y^{e_y-1} dt_y = t_y^{e_y-1} (wt_y + ue_y) dt_y.$$

As f is tamely ramified at y , e_y is invertible in k and $wt_y + ue_y$ is invertible, hence $v_y(c_y) = e_y - 1$.

- (c) Since f is wildly ramified at y , by the computation in (b), $c_y = wt_y^{e_y}$, hence $v_y(c_y) \geq e_y > e_y - 1$.
- (d) The morphism f is unramified at $y \in Y$ if and only if $(\Omega_f)_y = 0$. By (a), this is the case if and only if c_y is invertible, i.e., (c_y) is not contained in the maximal ideal $\mathfrak{m}_y$, i.e., $y \notin V(s_f)$.

□

Theorem 1.9 (Riemann-Hurwitz formula). *Let $f: Y \rightarrow X$ be a finite, generically separable morphism of curves over k . Then the genera of X and Y satisfy*

$$2g(Y) - 2 = n(2g(X) - 2) + \deg(R_f),$$

where $n = [\kappa(Y) : \kappa(X)]$ is the degree of f . Furthermore, if f is tamely ramified, then $\deg(R_f) = \sum_{y \in Y} (e_y - 1)$.

Proof. Since tensor products add degrees, whereas dualization negates the degree,

$$\deg(R_f) = -\deg(L_f) = \deg(\Omega_{Y/k}) - \deg(f) \deg(\Omega_{X/k}) = 2g(Y) - 2 - n(2g(X) - 2),$$

where the last equality follows from the computation of the degrees of the canonical divisors of Y and X .

Since Y is of finite type over k and k is algebraically closed, $[\kappa(y) : k] = 1$. Therefore, the second claim follows from Proposition 1.8. □

2. EXAMPLES

Definition 2.1 (Hyperelliptic curve). A hyperelliptic curve C over k is a curve of genus $g \geq 2$ over k with a finite morphism $\pi: C \rightarrow \mathbf{P}_k^1$ of degree 2.

Remark 2.2 (Affine charts via normalization). Hyperelliptic curves are given by equations $y^2 = f(x)$, where $f(x)$ is separable over k . Let $\pi: C \rightarrow \mathbf{P}_k^1$ be a hyperelliptic curve over k . Since π is of degree 2 and $\kappa(\mathbf{P}_k^1) = k(x)$, the function field of C is given by $\kappa(C) = k(x, y)$, where $y^2 = f(x)$ for some separable polynomial $f(x) \in k[x]$.

Since $V_0 = \text{Spec}(k[x])$ and $V_\infty = \text{Spec}(k[t])$ form an open covering of $\mathbf{P}_k^1$, where $t = x^{-1}$, the open subschemes $U_0 = \pi^{-1}(V_0)$ and $U_\infty = \pi^{-1}(V_\infty)$ form an open covering of C .

Furthermore, $U_0 \subseteq C$ is normal, which implies that $\mathcal{O}(U_0)$ is integrally closed in $\kappa(C)$. On the other hand, $\text{Spec}(k[x, y]/(y^2 - f(x)))$ is smooth over k , by the Jacobian criterion, hence $k[x, y]/(y^2 - f(x))$ is integrally closed in $\kappa(C)$ and $U_0 = \text{Spec}(k[x, y]/(y^2 - f(x)))$.

Let $n = \deg(f)$, $m = \lceil \frac{n}{2} \rceil$, $t = x^{-1}$ and $z = t^m y$ in $\kappa(C)$, with the relation $z^2 = t^{2m} f(t^{-1})$. By the same argument, we conclude that $U_\infty = \text{Spec}(k[t, z]/(z^2 - t^{2m} f(t^{-1})))$.

Example 2.3 (Ramification of a hyperelliptic curve). Let $\pi: C \rightarrow \mathbf{P}_k^1$ be a hyperelliptic curve over k . By [Remark 2.2](#), U_0 and U_∞ form an open covering of C . On U_0 , the morphism π is induced by the canonical map

$$k[x] \rightarrow k[x, y]/(y^2 - f(x)), \quad x \mapsto x.$$

A point of U_0 corresponds to $(a, b) \in k^2$ such that $b^2 = f(a)$. Since $x - a$ is a uniformizer of $\mathcal{O}_{\mathbf{P}_k^1, a}$, the ramification index at (a, b) is given by $e_{(a, b)} = v_{(a, b)}(x - a)$. Suppose that $f(a) = 0$, hence $b = 0$. By Euclidean division, there exists $h(x) \in k[x]$ such that $f(x) = (x - a)h(x)$, where $h(x)$ is invertible in the local ring $\mathcal{O}_{C, (a, 0)}$, as its image in the residue field is $h(a) \neq 0$. In this case, $x - a = h(x)^{-1}y^2$, hence $(x - a, y) = (y)$ and y is a uniformizer of the local ring $\mathcal{O}_{C, (a, 0)}$ and $e_{(a, 0)} = v_{(a, 0)}(x - a) = 2$. Conversely, if $f(a) \neq 0$, then there exists $h(x) \in k[x]$ such that $y^2 - b^2 = f(x) - f(a) = (x - a)h(x)$. As the image of $y + b$ in the residue field is $2b \neq 0$, it is invertible, hence $(x - a, y - b) = (x - a)$ and $x - a$ is a uniformizer of the local ring $\mathcal{O}_{C, (a, b)}$. In particular, $e_{(a, b)} = 1$.

On U_∞ , the morphism π is induced by the canonical map

$$k[t] \rightarrow k[t, z]/(z^2 - t^{2m}f(t^{-1})), \quad t \mapsto t.$$

It suffices to consider $t = 0$. Let $n = \deg(f)$ and $F(t) = t^n f(t^{-1})$, where $F(0) \neq 0$. In this case, t is a uniformizer of the local ring $\mathcal{O}_{\mathbf{P}_k^1, \infty} = k[t]_{(t)}$. If n is odd, then $z^2 = tF(t)$, so the unique point above $t = 0$ is $(0, 0)$. As $F(t)$ is a unit, $(t, z) = (z)$ and the ramification index at this point is 2. If n is even, then $z^2 = F(t)$ and the ramification index is 1 at both points.

By [Theorem 1.9](#) and the above calculations, we may compute the degree of the ramification divisor R_f as

$$\deg(R_f) = \begin{cases} n + 1 & n \text{ odd} \\ n & n \text{ even.} \end{cases}$$

Since $\deg(\pi) = 2$ and $g(\mathbf{P}_k^1) = 0$, the genus of C is computed as $g(C) = \lfloor \frac{n-1}{2} \rfloor$.

Lemma 2.4. *Let $f: \mathbf{P}_k^1 \rightarrow C$ be a dominant morphism of curves over k . If $\text{char}(k) = p$ and f is not generically separable, then there exists a dominant morphism $h: \mathbf{P}_k^1 \rightarrow C$ such that $\deg(h) = p^{-1} \deg(f)$.*

Proof. Since $k(t)$ is a $k(t^p)$ -vector space with basis $1, t, \dots, t^{p-1}$, the kernel of the universal k -derivation $d: k(t) \rightarrow \Omega_{k(t)/k}$ is $k(t^p)$. Consider the fundamental exact sequence

$$\Omega_{\kappa(C)/k} \otimes_{\kappa(C)} k(t) \xrightarrow{\eta^* df} \Omega_{k(t)/k} \rightarrow \Omega_{k(t)/\kappa(C)} \rightarrow 0,$$

where $\eta^* df(da \otimes 1) = da$. By [Remark 1.2](#), $da = 0$ for every $a \in \kappa(C)$, i.e., $\kappa(C) \subseteq k(t^p)$. This corresponds to a dominant morphism $h: \mathbf{P}_k^1 \rightarrow C$ of degree $\deg(h) = p^{-1} \deg(f)$. $\square$

Theorem 2.5 (Lüroth's theorem). *Let $f: \mathbf{P}_k^1 \rightarrow C$ be a finite dominant morphism of curves over k . Then $C \simeq \mathbf{P}_k^1$.*

Proof. By [Lemma 2.4](#), we may assume without loss of generality that f is generically separable. From [Theorem 1.9](#), the genus of C satisfies

$$-2 = n(2g(C) - 2) + \deg(R_f).$$

Since the ramification divisor R_f is an effective Cartier divisor, $g(C) = 0$. Clearly $C(k) \neq \emptyset$, hence it follows from the classification of genus zero curves that $C \simeq \mathbf{P}_k^1$. $\square$

Proposition 2.6 (Endomorphisms of curves of genus ≥ 2). *Let C be a curve over k such that $g(C) \geq 2$. Every finite, generically separable morphism $f: C \rightarrow C$ over k is an automorphism.*

Proof. Let $n = \deg(f)$. By [Theorem 1.9](#), the genus of C satisfies

$$0 = (n - 1)(2g(C) - 2) + \deg(R_f).$$

Since the ramification divisor R_f is an effective Cartier divisor and $g(C) \geq 2$, we must have $n = 1$. In other words, f induces an isomorphism of function fields, hence f is an isomorphism of curves. $\square$

3. TOPOLOGICAL GENUS

Throughout this section, a *surface* is a compact, connected, oriented topological surface without boundary.

Theorem 3.1 (Classification of closed surfaces). *Every surface S is homeomorphic to a sphere with $g_{\text{top}}(S)$ handles Σ_g , where the unique integer $g_{\text{top}}(S) \geq 0$ is the topological genus of S .*

Remark 3.2. In [Theorem 3.1](#), $S \simeq \Sigma_g$ admits a CW-decomposition with a 0-cell, $2g_{\text{top}}(S)$ 1-cell and a 2-cell, given by $4g_{\text{top}}(S)$ -gon decomposition. Therefore, the Euler characteristic is $\chi(S) = 2 - 2g_{\text{top}}(S)$.

Definition 3.3 (Topological ramification). Let $f: X \rightarrow Y$ be a map between surfaces.

- (a) f is a *branched covering* if it is surjective and locally of the form $z \mapsto z^{e_x}$ at every $x \in X$, where $e_x \geq 1$;
- (b) e_x is the *ramification index* of f at x ;
- (c) $x \in X$ is a *ramification point* and $f(x)$ is a *branch point*, if $e_x > 1$;
- (d) the *branch locus* of f is the set of branch points B_f in Y .

Ramification points are isolated, and a proper étale map is a covering map, hence the map $X \setminus f^{-1}(B_f) \rightarrow Y \setminus B_f$ is a covering and the following definition is valid:

- (e) the *degree* of f is the degree of the covering map $X \setminus f^{-1}(B_f) \rightarrow Y \setminus B_f$.

Lemma 3.4. *Let $f: X \rightarrow Y$ be a branched covering of degree n between surfaces. For every $y \in Y$, $\sum_{x \in f^{-1}(y)} e_x = n$. In particular, $|f^{-1}(y)| = n - \sum_{x \in f^{-1}(y)} (e_x - 1)$.*

Proof. Clearly, $f^{-1}(y)$ is a discrete closed subset of the compact space X , hence finite, i.e., $f^{-1}(y) = \{x_i \mid i \in I\}$, where I is finite. For every $x_i \in f^{-1}(y)$, choose open neighborhoods U_i of x_i and V_i of y such that $f(U_i) \subseteq V_i$ and $f|_{U_i} = z^{e_{x_i}}$. By the compactness of X , there exists some open neighborhood $y \in V \subseteq \bigcap_{i \in I} V_i$ such that $f^{-1}(V) \subseteq \bigcup_{i \in I} U_i$. Let $y' \in V \setminus B_f$. Write $f^{-1}(y') = \bigsqcup_{i \in I} (f^{-1}(y') \cap U_i)$. If y' is sufficiently close to y , then

$$n = |f^{-1}(y')| = \sum_{i \in I} |f^{-1}(y') \cap U_i| = \sum_{i \in I} e_{x_i}.$$

□

Theorem 3.5 (Riemann-hurwitz formula). *Let $f: X \rightarrow Y$ be a branched covering of degree n between surfaces.*

$$\chi(X) = n\chi(Y) - \sum_{x \in X} (e_x - 1).$$

Proof. Fix a finite triangulation of Y . Via refinement, we may assume that every branch point $b \in B_f$ is a vertex of the triangulation. Denote the sets of vertices, edges and faces by V_Y, E_Y, F_Y respectively. The connected components of the preimages of the open cells under f form a CW-decomposition of X . Since the interior of each face in Y is simply connected, its preimage consists of precisely n copies of 2-cells in X . Similarly, the preimage of the interior of each edge consists of n copies of 1-cells in X . By [Lemma 3.4](#), the number of 0-cells in X is given by

$$n|V_Y| - \sum_{v \in V_Y} \sum_{x \in f^{-1}(v)} (e_x - 1) = n|V_Y| - \sum_{x \in X} (e_x - 1),$$

where the last equality follows, because every branch point is a vertex of the triangulation of Y . We deduce the formula by computing the Euler characteristic of X via the CW-decomposition. □

Remark 3.6 (Topological genus and algebraic genus). Let C be a curve over $\mathbf{C}$. There exists a *complex analytic space* C^{an} associated to C , whose underlying set is the $\mathbf{C}$ -points of C . The smoothness and projectivity of C implies that C^{an} is a compact Riemann surface. A morphism $f: Y \rightarrow X$ of curves over $\mathbf{C}$ induces a proper, finite-to-one holomorphic map $f^{\text{an}}: Y^{\text{an}} \rightarrow X^{\text{an}}$ between compact Riemann surfaces.

Moreover, if $y \in Y(\mathbf{C})$ and e_y is the ramification index of f at y , the corresponding map f^{an} is given in suitable local holomorphic coordinates by $z \mapsto z^{e_y}$. Furthermore, the degree of f equals the degree of f^{an} .

Choose $t \in \kappa(C)$ transcendental over $\mathbf{C}$. The extension $\mathbf{C}(t) \subseteq \kappa(C)$ corresponds to a finite morphism $\pi: C \rightarrow \mathbf{P}_{\mathbf{C}}^1$. In fact, $(\mathbf{P}_{\mathbf{C}}^1)^{\text{an}} = S^2$, hence $g_{\text{top}}((\mathbf{P}_{\mathbf{C}}^1)^{\text{an}}) = g(\mathbf{P}_{\mathbf{C}}^1) = 0$. By [Theorem 1.9](#) and [Theorem 3.5](#), we conclude that $g_{\text{top}}(C^{\text{an}}) = g(C)$.

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