BLOCKSEMINAR
"LEARNING ALGEBRAIC GEOMETRY THROUGH CURVES"
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Let k be a perfect field. While a **curve** means just a separated integral projective scheme of finite type over k of dimension 1, other adjectives are often implied. For example, after quickly discussing normalization, we will talk about smooth curves[1]. Eventually we will also consider curves that are projective[2]. That being said, a word "curve" might have slightly different meanings below depending on the context.

1. Local structure of smooth curves, maps between curves

(1) Smoothness vs. Regularity: Jacobian criterion and Zariski tangent spaces.
Discuss the definition of a smooth k-algebra via the Jacobian criterion. Show that smoothness implies regularity.

(2) Smooth conics in $\mathbb{P}^2_k$: classification via non-degenerate bilinear forms ($\operatorname{char} k \neq 2$).
Recall that bilinear forms can be diagonalized, check the smoothness of $ax^2 + by^2 + cz^2 = 0$.

(3) Equivalent conditions for a Noetherian local domain of dimension 1: regularity, DVR, being integrally closed.
Sketch the equivalence of these conditions.

(4) * Dedekind domains: unique factorization into prime ideals.
Sketch the proof. Recall how Dedekind domains appear in number theory.

(5) Normalization of a ring and of a scheme.
Define normalization of an integral domain. Explain that it commutes with localization and "construct" normalization of an integral Noetherian scheme (or a k-variety).

(6) Normalization of a node and of a cusp.
Give some computations of normalizations, draw pictures.

(7) Normalization of a projective curve is a regular projective curve.
To show that the normalization is projective, one needs to check that it is finite, and that the finite morphisms are projective.

Possible references: [Vakil, "Normality and factoriality" (5.4)]

2. Closed points of smooth projective curves and dominant morphisms between curves

(1) Closed points are in one-to-one correspondence to discrete valuations.
Closed points yield discrete valuations by normality discussed in the previous section. To go from a discrete valuation to a closed point, one can use the projective embedding of the curve: show that for one of the affine spaces $\mathbb{A}^n$ covering $\mathbb{P}^n$ the affine coordinate ring of $C \cap \mathbb{A}^n$ is contained in the valuation ring, and the preimage of the valuation ideal yields then a closed point on the curve.
Possible references: [Hartshorne, I.6 "Nonsingular curves"], [Fulton, "Rational maps of curves" (7.1)].

(2) Separated integral varieties: uniqueness of extensions of morphisms.
Show that if X is a reduced scheme, Y is a separated scheme, then two morphisms $f, g\colon X \to Y$ agree on a dense open subset of X, then $f = g$. Note that projective schemes over k are separated.

[1] over a perfect field k regular curves and smooth curves are the same
[2] in fact, all proper curves are projective

Possible references: [Vakil, “Reduced-to-Separated Theorem” (11.3)], [Hartshorne, Chapter II, Exercise 4.2].

(3) Extension of morphisms to projective space in regular points of codimension 1.

Show that if X is an integral Noetherian scheme, p is a regular point of X of codimension 1, then any rational map $X \dashrightarrow \mathbb{P}^n_k$ defined on a dense open subset extends uniquely to a morphism on some open neighbourhood of p.

Possible references: [Vakil, “The Curve-to-Projective Extension Theorem” (15.3)], [Sha, “Propertiese of Nonsingular points”].

(4) Examples of extension of rational morphisms.
   (a) Let C be a smooth conic that contains a k-rational point c_0. Define a linear projection from c_0 onto a line in $\mathbb{P}^2$. Define its inverse and deduce that C is isomorphic to $\mathbb{P}^1$.
   (b) Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by $[x : y : z] \to ([x : z], [y : z])$.

(5) Dominant morphisms between curves and embeddings of field of functions.

Show that there is a contravariant equivalence between the following categories:
   (a) Smooth projective curves over k with non-constant (dominant) morphisms.
   (b) Finitely generated field extensions K/k of transcendence degree 1, with k-algebra field inclusions.

Possible references: [Vakil725, Classes 19-20].

3. (Pavel) Line bundles, linear systems, divisors, maps to projective spaces

(1) Description of maps to a projective space $\mathbb{P}^n$
(2) Line bundles and effective Cartier divisors, the Picard group
(3) From divisors to maps to a projective space: linear systems
(4) Examples: pencils of plane curves, relation to the quadratic Cremona map
(5) Weil divisors on smooth curves are Cartier
(6) Degree of a divisor on a curve and degree of the map

4. Maps from curves to a projective space and applications of the Riemann–Roch

(1) Relation between rational functions on a curve and global sections of line bundles.
(2) When does a linear system $|D|$ define an embedding of C to $\mathbb{P}^n$?
(3) Genus of a smooth projective curve. Relation to the genus of Riemann surfaces.
(4) Example: genus of a smooth plane curve of degree d. Degree of K_C.
(5) Formulation of the Riemann–Roch Theorem.
(6) Application: If $\deg D \geq 2g$, then $|D|$ is base-point free.
(7) Application: If $\deg D \geq 2g + 1$, then $\mathcal{O}(D)$ is very ample.
(8) Application: rational functions on a curve with prescribed zeroes and poles.

Possible references: [Vakil, “A criterion for a morphism to be a closed embedding” (19.1), ”A series of crucial tools” (19.2)], [Hartshorne, “Embeddings in Projective Space” (IV.3)].

5. Curves of genus 0

(1) Use the isomorphism of $x^2 + y^2 = z^2$ with $\mathbb{P}^1$ to describe the integral Pythagorean triples.
(2) T_C yields an embedding as a conic (Riemann–Roch).

Possible references: [Vakil, “Curves of genus 0” (19.3))]

(3) 3-dimensional quadratic forms, Clifford algebras, and quaternion algebras.

Construct a 1-to-1 correspondence between isomorphism classes of 3-dimensional quadratic forms and quaternion algebras as follows. Given a quadratic form (V, q), one constructs its Clifford algebra $Cl(q) = Cl_0(q) \oplus Cl_1(q)$, and if $\dim V = 3$, then $Cl_0(q)$ is a quaternion algebra. Given a quaternion algebra (or a central simple algebra of degree 2), one defines the subspace of trace-zero elements together with the reduced norm on it, which gives a 3-dimensional quadratic form.

(4) Maps between conics.

Let A, B be two non-isomorphic quaternion algebras. Prove that there are no maps between corresponding genus 0 curves.

(5) Tsen's theorem: every conic over $\overline{k}(t)$ has a rational point.

Application: let S be a smooth projective surface over $\overline{k}$ and let $\pi\colon S \to \mathbb{P}^1_{\overline{k}}$ be a dominant map such that its generic (or general) fiber has genus 0. Then S is birational to $\mathbb{P}^2$.

6. Riemann–Hurwitz formula: the Lüroth's theorem, endomorphisms of curves of genus $g > 1$

(1) Local ramification index.
(2) Ramification divisor and the main exact sequence.
(3) The Riemann–Hurwitz formula.
(4) Example: ramification for a hyperelliptic curve $y^2 = f(x)$.
(5) Application: Lüroth's Theorem.
Prove that if there is a dominant map $\mathbb{P}^1_k \to C$ to a curve C, then C is isomorphic to $\mathbb{P}^1_k$.
(6) Application: Every endomorphisms of a genus $g \geq 2$ curve is an automorphism.
(7) Application: genus is the topological genus over $\mathbb{C}$.
Explain how to prove the Riemann–Hurwitz formula for branched coverings of topological surfaces and the topological genus. Compare the genus defined in algebraic geometry to the topological genus.

Possible references: [Vakil, "The Riemann-Hurwitz Formula" (21.4)], [Hartshorne, "Hurwitz theorem" (IV.2)].

7. Cubic curves: construction of the group law

(1) Elliptic curves are cubic curves.
Use Riemann–Roch to embed a genus 1 curve with a rational point into $\mathbb{P}^2$.
(2) The group law on cubic curves.
Explain how to define the group law structure on the rational points of the smooth locus of the cubic curve. Prove that yields a functorial isomorphism $E(k) \cong \mathrm{Pic}^0(E)(k)$ for a smooth cubic E using Riemman–Roch.
(3) 2-Torsion points and the geometry of inflection points.
Identify 2-torsion points via geometrical data: p is a 2-torsion if the line connecting the zero point o to p is tangent to E. Compute these points for the Weierstrass equation $y^2 = x^3+ax+b$.
(4) Example: The nodal cubic and the multiplicative group $\mathbb{G}_m$.
(5) Example: The cuspidal cubic and the additive group $\mathbb{G}_a$.

Possible references: [Vakil, "Curves of genus 1"(19.9), "Elliptic curves are group varieties" (19.10)], [Silverman, "III.2 The Group Law"]

8. Elliptic curves: j-invariant, Jacobian, cohomology

(1) The universal property of an elliptic curve as its own jacobian.
(2) j-invariant of an elliptic curve as the ramification locus of the canonical map to $\mathbb{P}^1$.
(3) * Complex tori and elliptic curves: relation between lattices and j-invariants
(4) * Non-existence of the universal elliptic curve over $\mathbb{A}^1_k$

Possible references: [Hartshorne, "Elliptic curves" (IV.4)]

9. (2 Talks) Crash-course on cohomology, Riemann–Roch, and Serre Duality

(1) Cohomology of quasi-coherent sheaves: universal long exact sequences and functorial properties.
(2) Cech cohomology using affine coverings, proof of some properties of cohomology.
(3) Computation of cohomology groups $\mathrm{H}^i(\mathbb{P}^1, \mathcal{O}(d))$.
(4) Riemann–Roch theorem for curves
(5) Serre duality for curves

[Vakil725, `https://math.stanford.edu/~vakil/725/bagsrr.pdf`]

References

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[Hartshorne] Hartshorne, Robin. Algebraic geometry. Springer Science & Business Media.

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[Silverman] Joseph H. Silverman, ”The Arithmetic of Elliptic Curves”

[Sha] Shafarevich, Igor R. ”Basic algebraic geometry. 1. Varieties in projective space. Translated from the 1988 Russian edition and with notes by Miles Reid.” (1994).